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THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XVIII, Number 1

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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Volume XVIII, Number 1

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This bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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SUGGESTIONS FOR A MINIMUM HIGH SCHOOL MATHEMATICS COURSE*

By WARREN A. REES
Houston Junior College

After reading the title of this paper, you would be justified no doubt in expecting to find it consisting of lists of desirable and undesirable subject matter for a high school mathematics course. And for more or less obvious reasons it might have been wiser for me to have chosen that phase of my subject, for I am well aware of the lack of agreement, even among authorities, concerning methods of teaching; but for reasons which will follow I dare suggest that it is possible that we teachers of mathematics may improve our technique as well as the content of our courses.

In order that you may realize that I hold no illusions as to my ability to solve the problems I have suggested I quote the following:[1]

> The investigator of educational practices and methods of teaching is impressed with an unmistakable educational anti-climax; for the conviction grows on him that elementary school teaching is on a relatively high plane, that secondary school teaching is not as effective and that collegiate teaching, with rare exceptions, is ineffective and in urgent need of reform.

If this be true, what I have to say will be as applicable to college teaching as to any other, and from my limited experience and observation I fail to note any marked difference in the problems of teaching mathematics at various levels.

That there may be exceptions to the most widely accepted minimum essentials of good teaching is evidenced by the following situation. The head of the mathematics

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*Presented to the Mathematics Section, Texas State Teachers Association, Austin, Texas, November 30, 1933.

[1]Paul Klapper (Editor), *College Teaching*, p. 43. World Book Co.

department in a certain well-known mid-western university observes his instructors very carefully in the classroom and reports finding that, in all the classes of one instructor, all the students sat as near the back as possible and never uttered a sound from the time they entered until they left the classroom; the instructor faced the blackboard for a full hour and talked in a monotone as he wrote. The instructor was notified before the end of the year that he would not be retained. When the final examinations were given, which were departmental, every one of his sections led its respective group.[2]

Assuring you, if this be necessary after what we have just seen, that I hold no illusions of proposing a panacea for all the ills that beset us, I suggest the following as being minimum essentials for any course in mathematics.

No course in mathematics is complete unless a considerable majority of its students acquire a love or at least a wholesome respect for the subject. (I doubt if anyone would be so optimistic as to expect any ideal or goal in teaching to be attained by 100 per cent of his students.)

With this goal in mind, I am confronted with the suggestion from some of our students that mathematics is nothing but a set of mechanical rules for getting answers to certain problems; and that we as teachers of mathematics insist that they learn certain mechanical manipulations of words and symbols without regard to their meaning or possible future functioning in the lives of those whom we teach.

Of course I know that we say to ourselves that mathematical facts are neither useless nor meaningless to any save the uninitiated who fail to grasp their meaning or find a use for them.

If anyone doubts that there is an element of truth in their accusations I offer the following list of questions and problems and some answers and "solutions" which I am sure anyone who has access to enough of our ex-students can duplicate or improve upon.

---

[2]Joseph Seidlin, *A Critical Study of the Teaching of Elementary College Mathematics*, Columbia University Bureau of Publications.

1. State the meaning of a^m, where a and m are any real numbers and $a>0$. Answer: $a^m = a \cdot a \cdot a \cdot \cdots$

2. State the law of exponents for multiplication. Answer: When you multiply you add exponents.

3. Solve $\frac{2}{x} = 4$ for x. Answer: $x = 2$.

4. Simplify $\frac{\frac{1}{2} - \frac{2}{5}}{\frac{1}{2} + \frac{2}{5}}$. Answer: Almost anything you suggest, *e.g.*, 1, 2, 0, etc.

Or instead, visit my classroom or your neighbor's and listen to such questions as:

5. Do you multiply or divide in this problem?
6. What formula do you use?
7. Can you cross-multiply?
8. Can you cancel these numbers?

Or ask some student who has used logarithms to solve a triangle correctly to five digits, what the meaning of $\log_a N$ is, or to state the theorem used in finding the logarithm of a quotient.

Of course it is very confusing to the student if you make the following suggestions in connection with the above; *e.g.*, concerning No. 1, $(4)^{\frac{1}{2}} = ?$ No. 2, $a^2b^3 = ?$ No. 3, Check your answer. No. 4, How do you know? Is your answer reasonable? Or if, concerning No. 5, you point out that you can get the answer by either multiplication or division if you use a suitable factor of proportionality. Or if, concerning No. 7 or No. 8, you point out that you "can" do anything you please, and that what he probably wants to know is what the conclusion should be if he does a certain thing.

I am sure that you will all agree that any student who has been so unfortunate as to have acquired any of the incomplete or false ideas suggested above would not likely have much love or respect for the subject. As for his mathematics being of any use to him, that would depend on

whether or not his attempted practice necessitated his use of his pseudo-information.

Now if such fallacious notions do exist in the minds of our students, and I am sure I have mentioned only relatively few of the total, what can we as teachers and possibly administrators do about it? Frankly, I am not sure, but I suggest the following.

1. Let there be allotted sufficient time in the elementary grades for each student to master the four fundamental operations and problems related to them, and then introduce new ideas only as rapidly as it is possible for the student to get clear ideas of their use and meaning.

2. As early as possible in his career let each student learn that mathematical knowledge may be classified as follows:

(a) Undefined elements
(b) Defined elements
(c) Axioms
(d) Theorems

3. Let all students realize that mathematics is a logical science, and not confuse questions of whether one can or cannot do a thing with logical conclusions.

4. At the expense of economy in time let all statements in mathematics be complete sentences, even as the English department would suggest. And I am not sure but that this, combined with No. 3, is the crux of the whole situation. Certainly many common errors of the type pointed out above could and would be avoided by such practice. For example, anyone who had never said nor thought of the law of exponents for multiplication in any other way than: *any number with an exponent multiplied by the same number with an exponent is equal to the same number with the sum of the two exponents for an exponent,* would ever make such a mistake as $a^3 b^4 = (ab)^7$. Of course no one would deny the economy in using short words and symbols for long expressions and ideas, but if we overestimate our students' ability to make a true and complete transaction rapidly, we

lose in confusion what we attempt to gain by system. I would suggest frequent and rapid drill on meanings of abbreviated expressions if indeed it is not desirable to avoid them a great deal more than we do.

As another example, I find many students who will tell you immediately that the tangent of any angle is a over b, or, if more sophisticated, ordinate over abscissa, who when pressed for a more elaborate explanation finally admit they have no clear idea about what they say.

If I have, by any chance, arrived anywhere near one of our problems and its solution, I must add this further suggestion in order to be consistent.

5. That we concern ourselves more with a few things done well rather than too many things attempted superficially. This could be accomplished by substituting, to some extent, generalizations for mechanical rules.

6. Let the student realize that mathematics is neither dead nor static and that there are many things about it that neither he nor anyone else knows. For example, there are many periodicals devoted to new mathematics exclusively, and a suggestion of some well-known unsolved problems such as Fermat's Last Theorem might be worth while.

In conclusion, let me say that the ideas suggested above have been tried to some extent by a few teachers of my acquaintance and no doubt by many others. It is hoped that a more scientific and extensive test may be made in the future. Also, the author would be glad to communicate with anyone having any suggestions for further study or results from experience in using any of the ideas suggested above.

PSYCHOLOGICAL PRINCIPLES APPLIED TO THE LEARNING PROCESS AND TO TESTING IN MATHEMATICS*

By B. P. REINSCH
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Introduction

In this paper we shall recall a few elementary psychological principles and then show how they apply in the teaching of mathematics. Emphasis will be placed on definite applications to specific illustrations. To get suggestions on how to make our teaching more effective we must necessarily turn to psychology, because in all teaching we are concerned with the problem of training human minds. Psychology is the study of the human mind; it deals with the characteristics, peculiarities, and limitations of the mental processes; it gives us valuable information on the learning process in the form of certain guiding principles or laws. For good results, therefore, we teachers must make sure that all of our particular methods and practices conform to the laws of learning. Good teachers do this very thing unconsciously. We should all, however, make a more *conscious* effort to make the best possible use of the laws of learning. Let us check up! Possibly we are entirely disregarding certain important principles, or possibly we are not using them to the best advantage. As a basis for our discussion let us first state some of the important aims or objectives in our teaching.

Aims in Teaching Mathematics

Before we can decide on particular methods or practices we must know definitely what we are trying to accomplish. We must have clear-cut and well-defined ideas concerning our objectives. Exactly in what ways do we expect to train

*Presented at the meeting of the Texas Section of the Mathematical Association of America, Southern Methodist University, February 11, 1933.

the student's mind? What abilities and habits should be developed? Some important aims are the following:

A. *To help the student in the acquisition of knowledge.* We must help the student in the learning of a group of facts and principles, help him acquire an extensive vocabulary of mathematical ideas.

B. *To help the student understand thoroughly all fundamental principles.* The mere *knowledge* alone of principles is obviously of no value whatsoever unless accompanied by a thorough *understanding.*

C. *To develop the student's ability to use mathematics as a tool—to apply it.* The student should not merely be a mechanical manipulator or imitator. He should develop his ability to apply the principles of mathematics to new situations. It is well to note here that a student may have a thorough knowledge and understanding of principles, and yet be entirely incapable of using them in the applications.

D. *To develop the student's ability to think and reason correctly,* to systematically approach and analyze new problems, to do his thinking in an orderly manner. He should be trained to draw *correct* conclusions from given assumptions or given facts—not to "jump" to conclusions that might not be justified.

E. *To train the student to do independent thinking.* He should not follow slavishly the thoughts and developments of teacher or author. He should be encouraged to carry out demonstrations or investigations according to his own ideas, on his own initiative. He should be given plenty of opportunity to develop good judgment.

F. *To help the student appreciate the beauties and wonders in mathematics.* Occasional reflections or comments on the beautiful, the artistic, or the miraculous aspects of mathematics help to generate enthusiasm and interest.

G. *To help the student acquire the habit of alert and voluntary attention.* The teacher should have innumerable tricks at his disposal to encourage this habit.

H. *To help the student acquire the habit of a questioning attitude.* He should never accept as gospel truth all statements by author or teacher, but should be required to verify or check them. Get him out of the bad habit of taking things for granted. He should on his own initiative raise questions or state new problems. For example, encourage the student to change slightly the conditions in a given problem and to question himself as to the consequent changes in results. He might ask himself, "Under what

conditions does the problem have a solution, or have no solution?"

I. *To help the student to acquire the habit of persistently searching for relations, logical connections, and applications.* As will be shown later, this habit is very important for effective learning.

J. *To help the student acquire the habit of consciousness of principles.* The student should know what he is doing, and why he is doing it. If he works mechanically he is sure to make many errors. He can prevent many errors by justifying every step, by recalling carefully every principle used.

K. *To help the student develop the habit of self-activity.* He should not always be dependent on explicit instructions by the teacher. He should be encouraged to do things other than those expressly assigned. The student learns most from things he has worked out by himself. It will be noted that, in the foregoing statements of aims, emphasis has continually been put on *student activity.* The teacher is merely to be a director or guide for the student's activities.

Psychological Principles of Learning

A few of the elementary principles of psychology which govern in the learning process are listed below.

Thought and Memory

(a). *General Laws of Association govern all of our thinking and recall (memory).* All objects of thought are connected or associated with each other in various ways in our mind. We therefore recall objects and principles because of their relations to other objects and principles. There is no such thing as an isolated thought. The very idea of "thought" involves making associations, or connections, or comparisons. Some of the types of connections or associations are:

Association by Similarity. For illustration, seeing a certain person may bring to our mind another person whom he resembles. We recall the second formula of figure 1*a* from the first, because there is only a slight difference in sign. Similarly, the formulas of figure 1*b* present only a slight difference, the co-functions in one taking the place of the

Fig. 1 a) $\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$

b) $1 + \tan^2 x = \sec^2 x$
$1 + \operatorname{ctn}^2 x = \csc^2 x$

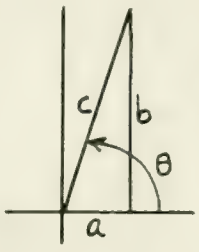
corresponding functions in the other. It is because of this same association by similarity that an elementary student makes the error in cancellation shown in figure 2*a*, which doesn't seem so very different from the case, figure 2*b*, where

Fig.2 a) $\frac{m+6}{6}$ b) $\frac{(m+6)6}{6}$

cancellation is permitted. Here the student has failed to make the *Association by Contrast, i.e.,* one can cancel factors, but one can't cancel terms. Similarly, in confusing $a^2 - b^2$ with $(a - b)^2$, the student notices a slight similarity but fails to make the necessary contrast: $a^2 - b^2 = (a - b)(a + b)$, while $(a - b)^2 = (a - b)(a - b)$.

Association by Cause and Effect. If we enter our home and find dresser drawer contents scattered about, we think immediately of robbers. The student can associate the value "zero" with ctn 90° by noting the effect of permitting the angle θ to approach 90°. (See Fig. 3.)

Fig.3



$\text{ctn}\,\theta = \frac{a}{b}$

As $\theta \to 90°$

$\frac{a}{b} \to \frac{0}{b} = 0$

Association by Whole and Part. Illustrations: A circle recalls to mind its radius, diameter, or circumference. A fractional exponent like 4 3 will recall the fourth power and the cube root. The expression $\cos^2 x - \sin^2 x$ should bring to mind $\cos 2x$, the whole of the formula $\cos 2x = \cos^2 x - \sin^2 x$ of which it is only a part.

Association by Substance and Property. A parabola should call to mind its property: a locus of points equidistant from a given fixed point and a given fixed straight line. A right triangle will suggest the Pythagorean theorem. Straight lines and parabolas are respectively associated with linear functions and quadratic functions.

(b). *Thorough understanding of and usefulness of a principle increase as more and more associations are made.* The degree of usefulness of a principle depends on the number and variety of applications (associations) that we have in mind. A fact or principle which is not attached to an application is worse than useless. For example, a student doesn't fully understand a "linear equation" until he

has used it repeatedly in the solution of practical problems in various fields. Why? More associations have been made. It is well to note that as more and more associations and applications of a certain principle are made, the student's true understanding of that principle increases. True understanding is a "growing affair."

(c). *Interesting things are remembered most easily.* Since the student learns most easily those things in which he finds pleasure, subject matter must be made interesting and appealing to him. Try to arouse his curiosity.

(d). *Extremely vivid things or experiences are easily remembered.* Forceful special examples and illustrations, good freehand blackboard sketches, make lasting impressions.

(e). *Easy to recall things that happened recently.* This principle shows the necessity for frequent reviews.

(f). *Easy to recall familiar things—things thoroughly understood.* After a thing is thoroughly understood the memorization will take care of itself.

(g). *Memory is aided by frequent reviews. New applications produce new associations.* Memory depends entirely on the association of ideas, the idea in mind recalling the ideas to which it has been linked by some past association. Each review will establish new associations and therefore improve the ability to recall. If in reviewing we apply principles to slightly new situations (new applications), then these old principles will be more thoroughly understood; and because more thoroughly understood they will be more easily recalled.

It is important to remember in this connection that the lesson that is studied but once is learned only to be forgotten. That which is repeatedly reviewed is intricately woven into the fabric of our thoughts and becomes a part of our permanent knowledge. We shouldn't forget that a student's true knowledge and ability are measured by what he *remembers permanently* and not by what he has once learned and recited. One fact well remembered is worth more than 100 facts which are forgotten.

(h). *Appeal to both the visual and the auditory memory.* It is very helpful to give graphical, geometric, and physical illustrations simultaneously with the algebraic, analytic, or abstract, thereby appealing to both types of memory. Many students have strong visual memories.

Habits and Abilities

(i). *To develop a habit, one act is not sufficient—constant repetitions are needed.* If the student is, for example,

to acquire the habit of always checking his work, it is not sufficient to tell him to do so once or twice. He must be required to do it always. Then too, the teacher himself should do it in his own board work in front of the students. The above teaching aims lettered G to K inclusive were intentionally stated as the development of certain desirable *habits.* This is to call attention to the necessity for constant practice and constant repetitions.

(j). *To develop certain specific abilities, we must constantly practice those very abilities.* A student who knows his laws of exponents for positive integers needs much additional practice in working with fractional exponents.

(k). *Mere practice does not make perfect.* The mere repetition of an activity does not improve it. Improvement depends on beneficial variations. This is nicely illustrated by the quadratic equation in algebra. In the sets of problems on the solution of quadratic equations many texts give twenty or more equations all in exactly the same form (like the first equation in figure 4), all monotonously alike, even

Fig. 4

$$2x^2-3x+5=0$$
$$4y^2=8y$$
$$3\omega^2=3-5\omega$$
$$bx^2+cx=ay$$
$$4R(1-R)=\sqrt{6}$$
$$.09t^2-3.2t=4$$
$$3x-\frac{1}{4x}+2=x$$

all using the same letter x. Here variation is needed. Many students who can easily solve the first equation will encounter difficulties if the equation appears in one of the other forms given in figure 4.

Particular Methods and Practices in Teaching Mathematics

We shall now suggest some particular methods of procedure which are justified by the foregoing Aims and are in harmony with the Laws of Learning. Immediately after each of the following statements (1), (2), (3), etc., we shall give the letters of the particular Teaching Aims (A, B, C, etc.) and the letters (a, b, c, etc.) designating the particular Principles of Learning listed above which apply to these

statements. The reader can then easily determine the exact reasons why these particular practices are justified or desirable.

(1). *Explain unknown facts in terms of the known facts.* (A, B, F, I, a, b, f). Since it is easier to remember related facts, be sure to tie up all new facts intimately with the known facts. Get as many relations and contacts as possible. Reduce the number of principles to a minimum and make others depend on these. This effects an organization of subject matter. Illustrations: The laws of logarithms should not be explained as new laws but rather as the old familiar exponent laws written (Fig. 5) in a new form. Similarly, the laws of radicals (Fig. 6) are nothing but the

Fig.5

Exponent Laws	=	Laws of Logs
$a \cdot b = c$		$\log c = x + y$
$10^x \cdot 10^y = 10^{x+y}$		$= \log a + \log b$

Fig.6 $\left(\sqrt[3]{5}\right)\left(\sqrt[3]{2}\right) = 5^{1/3} \cdot 2^{1/3}$
$= (5 \cdot 2)^{1/3} = \sqrt[3]{10}$

laws of exponents in a new form. The second identity in figure 7 is really the same as the first with each term divided

Fig.7 $\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$

by $\cos^2 x$. In analytics the derivation of standard forms of equations of straight lines, circles, parabolas, ellipses, etc., should all be considered as basically the same: all are problems of finding the equation of a locus. If the student is familiar with the graph of $y = \cos x$, he can easily sketch

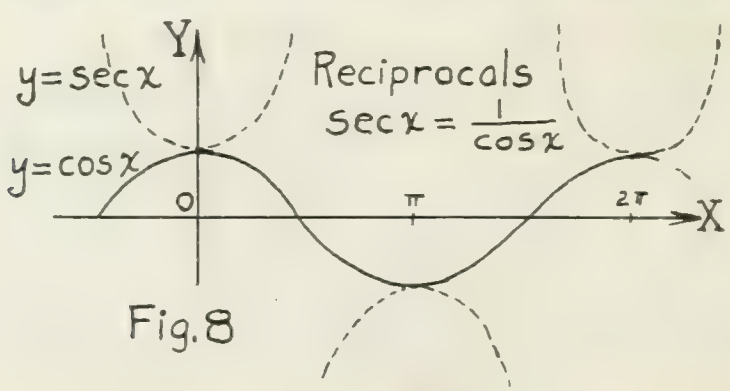


Fig.8

the graph of $y = \sec x$ (Fig. 8) because its ordinates are merely the reciprocals of the ordinates of $y = \cos x$.

(2). *Make pictures, sketches, or graphs wherever possible to illustrate.* (B, D, a, b, d, h). Rough sketches or graphs are of tremendous value in clarifying ideas, in creating interest, and in aiding recall. An idea is seldom so abstract that it can't be illustrated by some sketch or physical analogue. Students should always be required to draw figures or sketches for their problems. This will enable them to answer most of their own questions. Illustrations: Use trigonometric line values to help learn the fundamental identity connecting $\sin \theta$ and $\cos \theta$ (Fig. 9). The physical

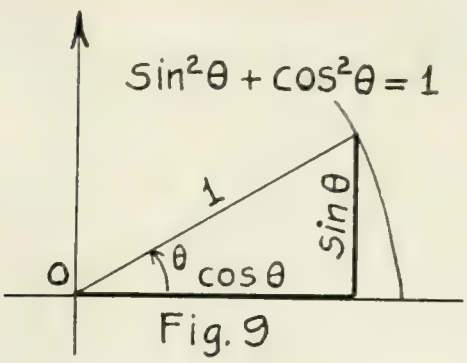
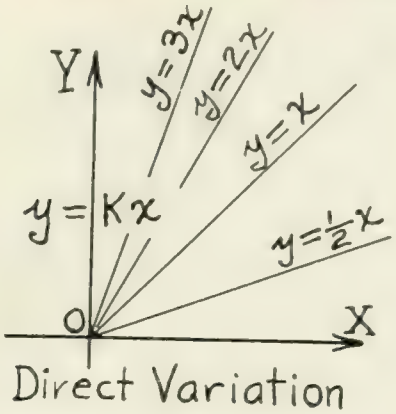
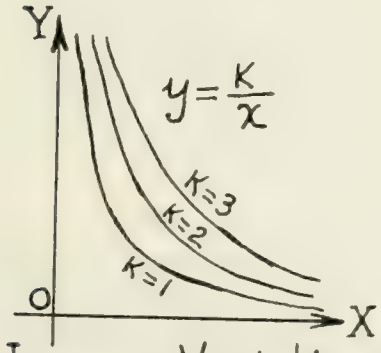


Fig. 9

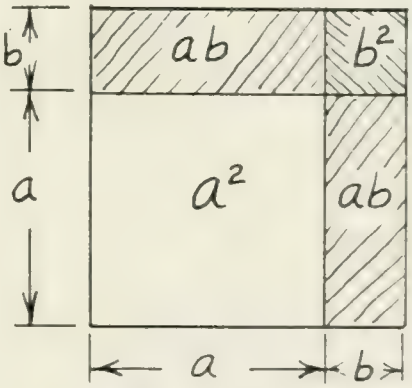


Direct Variation
Fig. 10

significance of direct and inverse variation is easily brought home to the student by sketches as in figures 10 and 11. Explain the formula giving the square of a binomial by means of areas as in figure 12. Explain the meaning of a



Inverse Variation
Fig. 11



$(a+b)^2 = a^2 + 2ab + b^2$
Fig. 12

differential of an area and the distinction between the differential and the increment of the area as in figure 13, and also graphically as in figure 14.

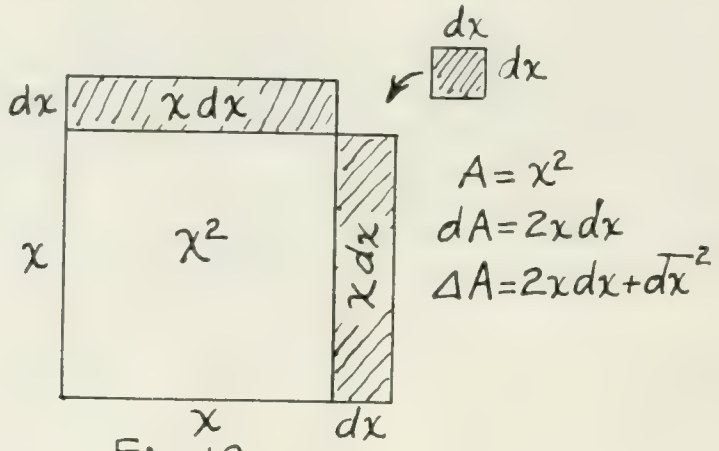


Fig. 13

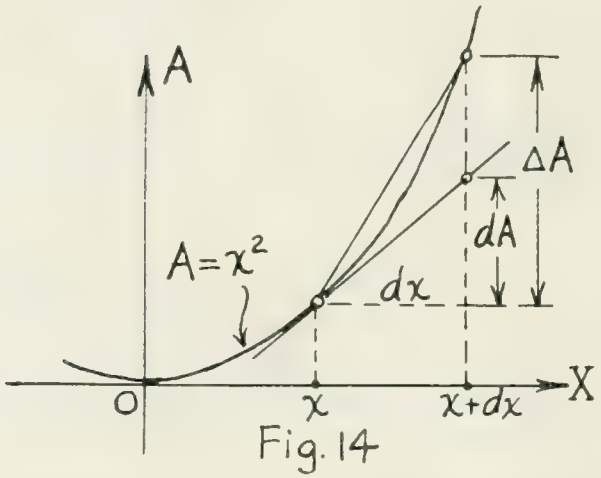


Fig. 14

NOTE: We are purposely selecting our illustrations from the entire list of elementary branches of mathematics to emphasize the fact that they all apply to each subject in particular. The reader can easily supply additional illustrations in each subject; for example, he can illustrate every principle many times in algebra alone.

(3). *Fundamental facts, principles, and methods should be recalled at frequent intervals and used in new connections.* (B, C, I, a, b, f, g). As already indicated, this enriches the student's understanding of principles, and aids his memory materially. The ability to solve a quadratic equation quickly under any conditions is so fundamental and important that students should be given every possible opportunity to work with it. Bring it in again and again wherever possible; when solving equations of higher degree

(Fig. 15*a*) ; again in exponential equations (Fig. 15*b*) ; and again when studying trigonometric equations (Fig. 15*c*).

Fig.15 a) $x^3+27=(x+3)(x^2-3x+9)=0$, b) $e^x+e^{-x}=4$, $e^{2x}-4e^x+1=0$,

c) $\cos^2 x+3\cos x=10$, $\cos x=?$ d) $3y=6-x-3x^2$ Find X-intercepts

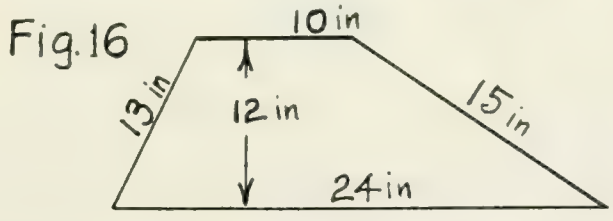
e) $x^2-xy-2y^2+3y-1=0$, from which

$$y=\tfrac{1}{4}\left(3-x\pm\sqrt{9x^2-6x+1}\right)=1-x \text{ or } \frac{x+1}{2},$$

f) $y=\frac{x}{x^2-3}$, $(y-1)^2x-y=0$

In analytics bring in the quadratic again in finding the intercepts of a curve (Fig. 15*d*) ; in identifying a curve or plotting a factorable locus like that in figure 15*e*; in finding asymptotes and discussing "extent" when equations like those in figure 15*f* have to be solved both for x and for y. Again in calculus bring in the quadratic equation wherever there is an opportunity. After such repeated applications a student is sure to know his quadratic. A similar plan should be followed in teaching other fundamental and important principles, for example: factoring, laws of exponents, elementary formulas from geometry, binomial expansion, etc.

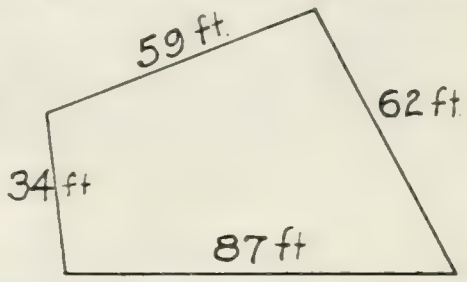
(4). *Often give extraneous, unnecessary, or inconsistent data.* (B, C, D, E, G, H, J). The student needs much practice in selecting the data required for certain problems, for this he must be able to do later in life. Give unnecessary data wherever possible, and occasionally inconsistent data. This encourages alertness and careful analysis. The mechanical manipulator will be exposed and penalized. Textbooks are very weak on this point. Illustrations: Ask for the area of a geometrical figure, like figure 16, in which more



than the necessary dimensions are given. Give students the value $\sqrt{2}=1.4142136$ when they need it correct to only three decimals. Find the equation of the straight line through the points $(-3,0)$, $(-1,1)$, and $(5,4)$, which is possible only if these points are collinear.

(5). *Occasionally give impossible problems or problems that have no solution.* (B, C, D, E, G, H, J, a, b, f). This encourages alertness and careful thinking. It is often just as valuable to know what can't be done as it is to know what can be done. Textbooks are deplorably weak on this point. Illustrations: Ask for the area of the quadrilateral in figure

Fig.17



17. Ask for the factors of expressions like those in figure 18 and bring out the idea of factorability in certain domains.

Fig 18

Factor:

$x^2-3=(x+\sqrt{3})(x-\sqrt{3})$

$x^2+4=(x+2i)(x-2i)$

$x^5+x+1=(\ ?\ )(\ ?\ )$

Ask for the solution of equations like those in figure 19, where equation (a) has no solution and equation (b) has

Fig 19

a) $1-x^2=x(5-x)-5(x+3)$

b) $x^2-24=x(x+8)-8(x+3)$

every number for a solution. Ask students to integrate the integrals in figure 20.

Fig.20 $\int\frac{dx}{\sqrt{1-x^3}}$, $\int e^{x^2}dx$

(6). *Teach all the different types of notation. Use them all frequently.* (A, B, C, a, i, j). For example, all the different symbols for ordinary and partial derivatives, figure 21.

Fig.21 $\frac{dy}{dx}$, $\frac{d}{dx}(\ )$, $\phi'(x)$, $\phi''(x)$

y', ϕ_x, $\dot{x}=\frac{dx}{dt}$, $\ddot{x}=\frac{d^2x}{dt^2}$

(7). *Change the set-up and letters in proofs and demonstrations.* (B, C, J, f, j). Make the students independent of notation or particular letters. If the text uses the letters *A*, *B*, and *F* on a figure, then have the student use others like *P*, *Q*, and *R*. If the text gives a certain demonstration with an acute angle, then require the students to work it out using an obtuse angle. Change the conditions or assumptions in some way. Reverse figures, or turn them upside down.

(8). *Create interest by humanizing the subject.* (A, F, a, b, c, f). Bring in the human element, little bits of history, illustrative personal anecdotes, or special applications, wherever they enter in a natural way. Illustrations: Napier and Briggs, logarithms; Gauss with the many different proofs for the fundamental theorem of algebra; Abel, young Norwegian, age 27, fifth degree equation; Fermat's last theorem.

(9). *Give students practice in setting up problems.* (B, C, D, E, G, H, I, J, K, b, d, j, k). State problems so that students are required to determine for themselves what particular methods, formulas, or relations are needed. This is required in all applications in daily life. For illustration, ask for the area of a triangle with sides 9 feet by 9 feet by 3 feet. The student will then have to discover that this is an isosceles triangle and decide on the proper method of attack. Areas of composite figures like figure 22 give good practice. In asking for the computation by integration of the centroid of any right circular cone do not give any equations or coördinate axes; let the student determine these.

Fig.22

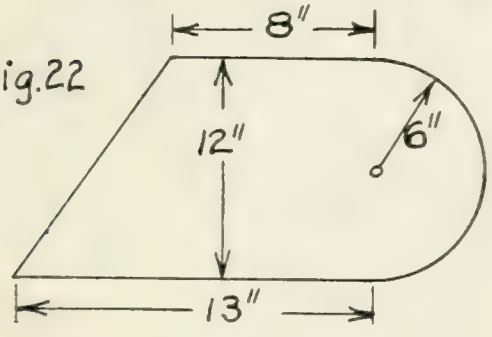
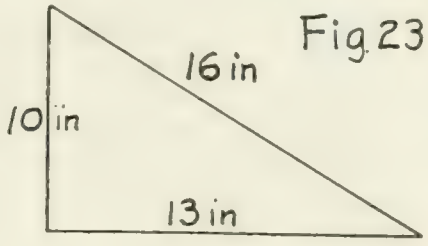


Fig.23



(10). *Frequently confront the student with unusual situations and problems.* (B, C, D, E, G, H, J, a, b, c, d). Illustrations: Ask students to find the area of a triangle as in figure 23 without telling them that it is or is not a right triangle. Since this looks like a right triangle many will erroneously assume that it is, without first testing. Find

the solutions of equations in figure 24. Ask for three solutions of the equation $x - 2y = 6$. While studying graphs of linear equations ask the student to sketch $y + 6 = 3x$.

Fig.24 Solve: $(x+\frac{2}{3})^2 = \frac{3}{9}$

$x + \frac{1}{5} = \frac{3}{5}\sqrt{3}$

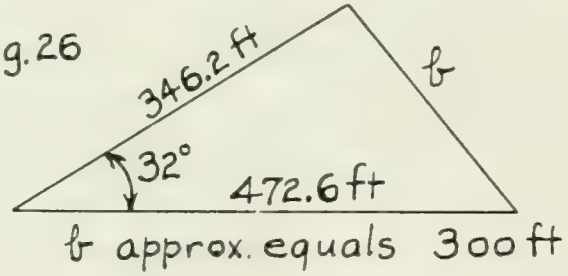
Fig.25 $y^2 = 4x^2$

$(x-3)^4 + (y-2)^4 = 4$

Some will erroneously treat this as a linear equation. Similarly, when studying parabolas and circles ask for graphs of the equations in figure 25. Many will erroneously draw a parabola and circle respectively, and consequently show that they need to learn much more about these standard curves.

(11). *Teach students self-reliance by requiring preliminary estimates, checks, alternate methods of attack, tests of reasonableness of result.* (C, D, E, H, K, a, b, d, f, j, k). Before solving the triangle, figure 26, draw the triangle approximately to scale and estimate the length of the side

Fig.26

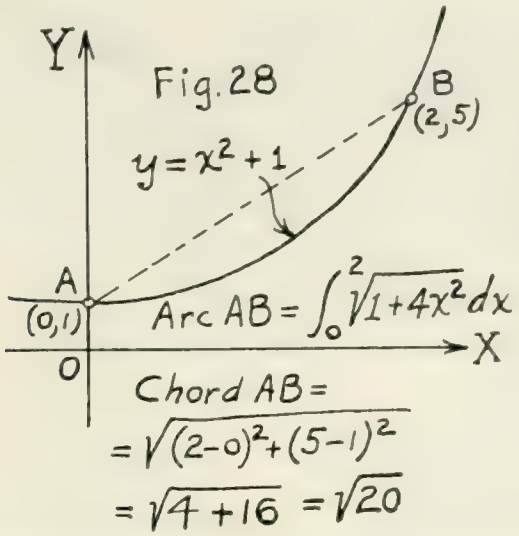


"b," which seems to be about 300 feet. (It is surely not over 400 feet nor less than 100 feet.) Rough approximations as indicated in figure 27 serve nicely to check the po-

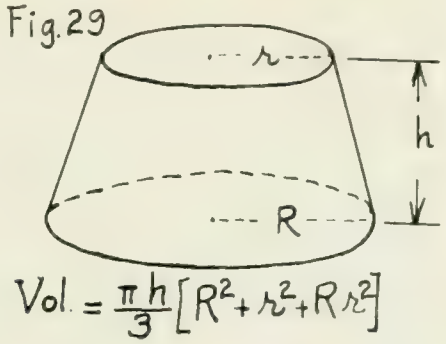
Fig.27

$$m = \sqrt{(3.86)(53.34)} = 14.4 \text{ (accurately)}$$
$$= \sqrt{4(50)} = \sqrt{200}$$
$$= 14 \text{ (approximately)}$$

sition of the decimal point. The value of the definite integral in figure 28, giving the length of the arc AB, can be roughly checked by computing or measuring the length of the chord AB. Encourage students to work problems by two or more methods, one result checking the other. Often general formulas or relations can be checked by the homogeneity test: all terms must be of the same dimensions. For



example, in figure 29, the formula for the volume is obviously incorrect because the last term in the parentheses is of dimension three instead of dimension two.



(12). *Give students opportunities to make errors.* (B, C, D, G, H, J, a, d, i, j). This is very important in testing the student's true understanding of principles. A good plan is to keep a list of errors commonly made and then design problems containing temptations to make these errors. This is best illustrated by the simplification problems in figure 30 and by the equations to be solved in figure 31.

Fig. 30 Simplify

$(4x^{3/2})^2$, $\frac{R^{4-2x}}{R^2}$, $\frac{27^{20}}{3^{10}}$, $\frac{4-\frac{1}{4}m}{m^2}$,

$\sqrt{\frac{1}{4}-\frac{1}{x^2}}$, $\sqrt{a^2+a^2m^2}$, $\sqrt{10^{3.26}+10^{4.32}}$,

$\frac{\sqrt{5}}{\sqrt{5}-5}$

Just try these on your students. If a student violates any principles in solving these, I would say most emphatically that he does not yet understand those principles as he should.

(13). *Don't teach simplest illustrations of principles only.* (A, B, C, a, b, f, j). Always bring in a few more complex problems. In late years many textbooks have sim-

Find all the solutions of:

Fig.31 $x^3=8$, $4x(1-x)=1$, $x^2(x-2)=5(x-2)$, $25^x-5^x=30$.

Factor:

Fig.32 $(1-x)^4-(1-x)^2$, x^3-x^2-x+1

$1+x^3+(1+x)^3$, $3x^2(1-x)^{3/2}+6x\sqrt{1-x}$

Fig.33 $$\frac{d}{dx}\left(\frac{x^2}{\sqrt{a^2-x^2}}\right)=\frac{2x(a^2-x^2)^{1/2}-x^2\left(\frac{1}{2}\right)(a^2-x^2)^{-1/2}(-2x)}{a^2-x^2}=\frac{(2a^2-x^2)x}{(a^2-x^2)^{3/2}}$$

plified all problems and illustrations to the point of making them unnatural. Such treatments are often harmful in preventing students from getting a complete and thorough understanding of principles. For example, all problems on linear and quadratic equations are often too simple. In factoring bring in more advanced problems as illustrated in figure 32, of which the last one represents a type frequently encountered in calculus. In figure 33 we have an ordinary type of problem in the calculus involving algebraic simplifications, showing why students in calculus are often handicapped because their algebra course took up only the very simplest types of simplifications.

(14). *Train the speed of perception by appropriate drills.* (C, G, e, g, j, k). Give plenty of practice in the speedy recognition of types or relations, in the determination of proper methods.

(15). *Students learn by doing and not by being told.* (B, C, D, E, G, I, J, K, a, b, d, j). Check up continually to make sure that the student does the required work. In attacking new problems the teacher can work out a problem on the board while the student should be required to go through the same work himself at his seat—the teacher's blackboard work always staying one step behind that of the student.

(16). *Frequently give problems embodying two or more different principles—small research problems.* (B, C, D, E, G, H, I, J, K, a, b, c, g, j, k). This is exactly the way the

fundamental principles of mathematics are to be used in the applications. Research ability is developed gradually, and should be practiced in an elementary manner as soon as possible. For example, ask for the diameter of a steel ball weighing one ton. Ask the student to find the length of AB in figure 34. To do this he must first compute certain other elements, carefully planning his work. A fine illustration is the following: Ask the student to prove that *any*

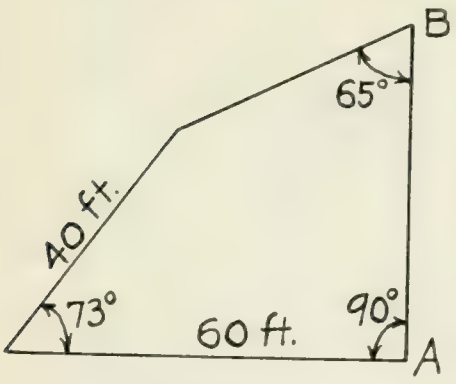


Fig. 34

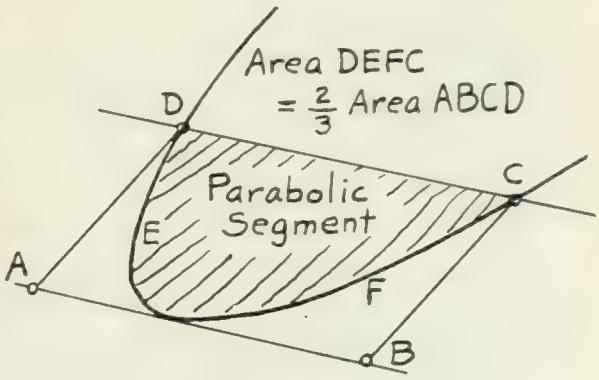


Fig. 35

segment cut from *any* parabola has an area two-thirds that of the circumscribed parallelogram; see figure 35. He will first need to decide on positions of axes and equations of curves, and then use algebra, analytics, and calculus.

(17). *Give problems that first require certain necessary simplifications or adaptations before proceeding with the regular solution.* (C, D, E, G, H, J, K, a, b, e, f, g, i, j, k). Illustrations: In figure 36*a*, each radical must first be put into a simpler form. The expression in figure 36*b* should

Fig. 36

a) Add $\sqrt{28} + \sqrt{\frac{9}{2}} - \sqrt[4]{9}$

b) Evaluate $\frac{\csc^2 x - \cot^2 x}{\csc x}$ for $x = 27°$

c) $\frac{d}{d\theta}\left(\frac{\sin 2\theta}{\sin^2\theta - \cos^2\theta}\right) = \frac{d}{d\theta}(-\tan 2\theta)$

first be simplified to sin x. In figure 36*c*, before differentiating the quotient, simplify it as indicated. By means of such problems the student is made to realize that he can save much work and time by first looking for simplifications

or changes. He also learns the value of being able to recognize certain trigonometric identities at a glance.

(18). *Explain or verify general principles by means of particular cases or illustrations.* (B, I, a, b, d, f). For example, demonstrate the error in figure 37 by simple num-

Fig. 37

$$\sqrt{a^2+b^2} \stackrel{?}{=} a+b$$

$$5=\sqrt{25}=\sqrt{9+16} \stackrel{?}{=} 3+4$$

$$=7$$

Fig. 38

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

True if $\theta = 30°$

$$\frac{1}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

bers as indicated. The truth of identities as in figure 38 is made convincing by verification as indicated.

(19). *Encourage careful and accurate thinking in various ways.* (D, G, i, j). For example, distinguish carefully between "equal to" and "approximately equal to." If certain formulas require all dimensions in the same units, then give some of the data in feet and some in inches.

(20). *Distinguish between learning verbal rules and definitions and learning facts.* (B, H, b, f). Rules and definitions are often misused. One could teach an eight-year-old child to recite correctly the formula for the area of a trapezoid without his knowing what a trapezoid is; or even to give the correct definition of "function." Obviously the mere reciting of rules and definitions has no connection with the knowledge of or understanding of facts. A rule should be the result of a set of experiences. It should never be a substitute for experience. Learn the facts first, and then the main features or significant ideas might be summarized in a rule. Questions should always aim to test the understanding of principles rather than call for their reproduction from memory.

(21). *Give data with specified units and then require results in correct units.* (B, D, j). The practice of some mathematics texts of often giving data or dimensions as pure numbers with no definite units attached should be condemned. Students need much practice in working with various units. For example, give the dimensions of a solid in inches and require the volume in cubic feet. Again, if the student is required to find the kinetic energy of rotation of a certain sphere in foot-pounds, some dimensions should be given in inches, possibly the density in pounds per cubic inch, and the angular velocity in revolutions per minute,

say. Then the student must change all of these to the correct units to get the result in foot-pounds.

(22). *Bring in decimals wherever possible and natural. Don't use integers exclusively.* (C, j, k).

(23). *Frequently give short and long examinations of the correct character to compel the student to organize his material and carry it in mind.* (A, D, I, b, g, j). The making of tests and examinations is a science which requires constant study.

(24). *Comprehensive examinations covering several subjects, or two or more years of work, are very desirable.* (A, B, C, D, J, K, a, b, g, i, j, k). These can be made very helpful in requiring students to organize and find relationships between related subjects. They combat the tendency to permit a subject to drop out of mind as soon as it is no longer required for a time.

(25). *Continually establish relationships and contacts between the subject you are teaching and related subjects.* (B, C, I, a, b, g, j). We are not permitting our laws of learning to help us when we treat all of our subjects too strictly in separate compartments. When teaching geometry, bring in arithmetic and algebra wherever you can. When teaching calculus, bring in all the geometry, algebra, trigonometry, and analytics you can. A teacher of analytics or calculus should expect to teach his students some more algebra. Remember that all knowledge is a "growing affair." This coördination of material is of inestimable value to the learner. As an extreme illustration, a mathematics teacher should correct gross errors that students may make in English. It is certainly very valuable and desirable to make as many contacts as possible between mathematics and the physical sciences. To be able to do this, a teacher will need to study related subjects continually.

(26). *Measure the achievements of a student not by grades made on the old essay-type examination, but rather design new type tests to determine their true understanding of principles.* (B, C, D, E, J, a, b, j). Examinations should very rarely merely call for things the student has memorized.

Conclusion

May this outline of suggested methods of putting our teaching on a good psychological basis urge all of us to do some real research in effective methods of teaching.

We teachers of mathematics have wonderful opportunities to train the student's mind, to develop his reasoning powers. Are we doing this as effectively as we should? Doesn't a thorough study of psychology and the learning process show us that we are far from being as scientific in our teaching methods as we are in other things?

Much of the current criticism by the general public of the results of our school work is probably due to the lack of emphasis on the part of teachers on aims lettered "B" to "K" as I have listed them. Too many concentrate 95 per cent of their attention on the aim labeled "A" and fallaciously assume that all the other aims will take care of themselves. Each time a teacher designs his "lesson plan" or constructs a "test," he should recall all of these psychological principles and attempt to make the best possible use of them. In systematically applying these principles, a teacher will constantly discover how he can improve his methods. Yes, teaching is a man's-size job. Above all, teaching is a science.

SOME FUNDAMENTAL DEVELOPMENTS IN MATHEMATICS*

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When it was suggested to the writer of this paper that he prepare for presentation before the Mathematics Section of the Texas State Teachers Association a paper with the above title or one similar, he decided that the purpose of the paper should be to arouse the interest of his auditors in further studies in mathematics by describing a few of the fundamental developments in mathematics. He was therefore confronted with this problem of selection: to choose topics which are fundamental and which are directly related to those branches of mathematics in daily use by his auditors. At the same time it was decided to make the discussion purely qualitative. The following four topics were selected:

1. The arithmetic definition of the irrational number.
2. The Non-Euclidean geometries.
3. The group concept.
4. Analysis situs.

These topics will be discussed in the above order. Restrictions of space and time preclude any but the briefest discussions.

1. *The Arithmetic Definition of the Irrational Number*

We have all encountered irrational numbers in the course of our mathematical experience. When we think of an irrational number, we usually give $\sqrt{2}$ as an example. When an irrational number appears in a computation, we approximate to the number of decimal places required by the allowable percentage of error and proceed according to the usual laws. But just what is an irrational number? To

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answer this question we propose to outline the arithmetic definition of the irrational number. By the arithmetic definition of the irrational number we mean a definition based only on the operation of counting, with no reference to any idea of measurement. Counting is an exact process. On the contrary, any physical measurement is at best an approximate process, and ideal measurements involve a considerable amount of abstraction. Hence we first examine the process of counting.

Counting can be carried out by a being to whom discrete objects, physical or ideal, are presented, and who possesses the following fundamental notions:

(1) The notion of *unity, i.e.*, the capacity to conceive of the form under which an object, physical or ideal, is regarded as a single one. The complexity of the object is unimportant so long as it can be distinguished from other objects.

(2) The notion of a *collection* or *aggregate* of objects, or *class* of objects, which can be conceived of as having more or fewer objects. An aggregate of objects is not merely a group of objects to each of which unity is ascribed, but is itself an object to which unity is ascribed when it is regarded as a whole. The individual objects which make up the aggregate are called the *elements* of the aggregate.

(3) The notion of *order, i.e.*, the capacity to assign relative rank to the elements of an aggregate, so that the aggregate becomes an ordered aggregate. This order may be assigned in various ways. However, order is itself an abstract conception independent of any mode of ordering. For an aggregate to be an ordered one it is necessary that each element possess a rank, in virtue of which it is known relative to any two elements which has the higher rank.

(4) The notion of *correspondence, i.e.*, the capacity to associate two aggregates so that to each element of the one there corresponds a definite element of the other. The correspondence may be complete in the sense that to each element of either aggregate there corresponds one and only

one element of the other; the correspondence may be incomplete in the sense that one of the aggregates may have one or more elements to which correspond no elements of the other. In this latter case we say that the aggregate with the superfluous elements contains more elements than the other, and that the latter contains fewer elements than the former.

The origin of these notions is something for psychologists and philosophers to discuss. We are not now interested in this question. It is sufficient for us that this set of notions form a basis for the process of counting.

We now briefly examine the ordinal number. Our starting point is the idea of the finite ordered aggregate. An ordered aggregate is said to be *finite* if

(1) There is one element which has lower rank than any of the others.

(2) There is one element which has higher rank than any of the others.

(3) Every part of the aggregate has an element which has higher rank than any other element of the part, and an element which has lower rank than any other element in the part.

The last condition is equivalent to the statement that every part of a finite aggregate is also a finite aggregate.

Two finite ordered aggregates are said to be *similar* if (1) to each element of the one there corresponds one and only element of the other, and conversely; (2) to any two elements P, Q of the one there correspond two elements P', Q' of the other having the same relative rank as P and Q. If two finite ordered aggregates are similar, they are said to have the same *ordinal number*. An easy process of reasoning shows us that if each of two finite ordered aggregates is similar to a third finite ordered aggregate, they are similar to each other. Hence we may regard an ordinal number as characteristic of an aggregate of similar ordered aggregates.

An aggregate consisting of the single element A is given the ordinal number 1. If to A we add an element B and

regard B as being of higher rank than A, we obtain the aggregate having the ordinal number 2. This process is continued to form the ordinal numbers 3, 4, 5, ... Every ordered aggregate formed in this manner is a finite ordered aggregate. It can be shown that the ordinal number of a finite ordered aggregate is not changed by re-ordering the aggregate in any manner.

Two aggregates are said to be *equivalent* if a one-to-one reciprocal correspondence can be established between their elements. This property of equivalence differs from the property of similarity in that no reference is made to the idea of order. We now say that an unordered aggregate is finite if it can be ordered so as to become a finite ordered aggregate in the sense already defined, and further say that two finite equivalent aggregates have the same *cardinal number*. Two aggregates which are each equivalent to a third aggregate are equivalent to each other. Hence a cardinal number is characteristic of a class of equivalent finite aggregates. The finite cardinal numbers are denoted by the same symbols 1, 2, 3, ... as are the finite ordinal numbers. However, the two classes of numbers are logically distinguishable.

The operations of addition and multiplication follow from the manner in which we have introduced the cardinal and the ordinal numbers. These operations are always possible, but this is not true for the operation of division, as the quotient is only in special cases an integer. Hence it is necessary to extend our number system so that division is always possible. Thus appear the fractional numbers. Historically these arose from the ideas involved in measuring. Since we want an arithmetic definition for the fractions we shall have to abandon the idea of measurement. The fraction is regarded as an association of a pair of integers, the associated integers being regarded as a single object. Laws for operating with the fractions are necessary. These laws are formulated in accordance with a principle known as the Principle of the Permanence of Forms.

This principle has four main parts. If we now understand by the unextended number system the sets of the cardinal and the ordinal integers, these four parts are: (1) every operation represented by a formal expression involving the unextended system of numbers and not resulting in a number of the unextended system, must have assigned to it a meaning of such a character that the formal expression may be handled as if the formal expression represented a member of the unextended system; (2) an extended definition of number must be given such that a formal expression of the kind considered in (1) represents a number in the extended sense; (3) a proof must be given that the same formal laws hold for the extended system as for the unextended system; (4) definitions of the terms "greater," "equal," "less" must be given in the ordinal sense.

Thus the association of two integers called by us a fraction is added to the number system. Our number system now consists of the positive rational numbers. Our next step is to introduce the negative numbers arithmetically.

The operation of subtraction, the operation inverse to that of addition, is at this stage only possible when the minuend is greater than the subtrahend. We now examine closely this operation, and formulate its properties. These properties, in accordance with the principle of permanence, serve to define the negative numbers. This method of introducing the negative numbers leads to their characteristic properties, such as the law of signs for the operation of multiplication. When we introduce the negative numbers, the number 0 naturally appears. It may be considered as characteristic of an empty aggregate, that is, an aggregate containing no elements.

We now consider the irrational numbers. First we consider a given rational number N. This number N divides the class of rational numbers into two classes: all rational numbers lower in rank than N and all rational numbers greater in rank than N. If N be assigned to the first class, there is no least element in this class and N is the greatest number of this class. If N be assigned to the second class,

it is the least number in this class and there is no greatest number in this class. The number N is said to define a *cut* in the set of the rational numbers.

It is possible to give examples of cuts in the system of rational numbers to which correspond no rational numbers. For example, consider a positive rational number M which is not a perfect square and the two classes of rational numbers: (1) all of the negative rational negative numbers and those positive rational numbers the squares of which are less than M; (2) all positive rational numbers the squares of which are greater than M. Such a number M defines a cut to which corresponds no rational number. There is no greatest number in the first class and no least number in the second class. Such a cut by definition gives a number which we call the square root of M. This example illustrates the manner in which the irrational numbers are fitted into our number scheme.

The idea of a cut, or, as it is sometimes called, a *section*, was introduced by the mathematician Richard Dedekind. The great German mathematician Georg Cantor defined the irrational numbers by means of convergent sequences of rational numbers. His definition is completely equivalent to the definition of Dedekind, and has the advantage of greater convenience in constructing these numbers.

The arithmetic definition of the irrational numbers has been a task fraught with extreme difficulty. The theory is nevertheless extremely interesting, and its study will well repay any person interested in mathematics. Anyone interested will find material for reading in any good work on the theory of functions of real variables.

2. *Geometry*

Geometry is that branch of mathematics, broadly speaking, which attempts to describe the space in which natural phenomena occur. Unless long training has broken us of the habit, most of us think of the exposition of Euclidean geometry given in the elementary texts when geometry is mentioned. This question occurs to us: Is space a configuration which has an independent existence and imposes

its mode of description on us, or is space a creation of our minds, and thus susceptible to different modes of description? It is our purpose to show that the latter viewpoint is correct, and that to think of Euclidean geometry as the only geometry is false. No disparagement of Euclidean geometry is implied in the latter statement; all that we wish to convey is that Euclid has no monopoly in the field of geometry.

We might effect a classification of geometries according as we do or do not admit the idea of measurement. In the latter case we are limited to the positional properties of geometric figures. We call the geometry which treats of such properties *projective* geometry. As its name indicates, *metric* geometry includes all branches of geometry in which appears the idea of measurement. When we consider the space in which natural phenomena occur, we utilize almost altogether a metric geometry for its description.

A geometry usually goes through two stages during its development. The first stage is a gradual accumulation of a body of propositions or theorems and their applications. The second stage consists of systematically classifying and grouping these theorems and subjecting them to a critical examination. This stage of development is really an examination of the foundations of the geometry. A set of undefined terms is constructed, this set naming the elements out of which the configurations of the geometry are constructed. A set of propositions which are accepted without proof is laid down concerning these undefined terms. These propositions are called *axioms*, or assumptions, or better, *postulates*. These postulates must be consistent among themselves and as few in number as possible, *i.e.*, the axioms must be independent.

Euclidean geometry went through these two stages of development. It probably had its earliest beginnings among the pyramids and irrigation ditches of ancient Egypt. Its propositions accumulated throughout several centuries until a man with sufficiently sharp mental abilities came along to carry it through the second stage. This man was Euclid.

Among the postulates laid down by him was one to the effect that, given a line L and a point P not on L, there can be drawn through P in the plane of P and L one and only one line not meeting L. This postulate was the object of the closest scrutiny on the part of Euclid's earliest commentators, the question being whether this postulate could be derived from the other postulates.

This question occupied the attention of a great many competent mathematicians until the decade 1820–1830, when the problem was solved by the Russian mathematician Nicholas Lobatchewsky and the Hungarian mathematician Johann Bolyai. Their procedure was simply to replace the parallel postulate by one incompatible with it, and then deduce a logically consistent geometry. They assumed that through P there could be passed an infinitude of lines in the plane of L and P not meeting L. The researches of Lobatchewsky and Bolyai marked the beginning of the non-Euclidean geometries.

The next high point in the development of non-Euclidean geometry came in the year 1868, with the posthumous publication of Riemann's paper, "On the Hypotheses Which Lie at the Foundations of Geometry." G. F. B. Riemann (1826–1866) was a professor of mathematics in the University of Goettingen, Germany, and is one of the immortals of mathematics. He was a student of Gauss, also a professor at Goettingen and another one of the great mathematicians of all time. Riemann received a great deal of inspiration in the formation of his geometry from Gauss' theory of surfaces.

We now examine hastily this geometry of Riemann. The first point that we notice is that Riemann regarded all spaces as only special examples of n-dimensional manifolds. An *n-dimensional manifold* is a set of points each of which is determined by an ordered set of numbers called *co-ordinates.* Next we notice that Riemann's geometry is what we call an *infinitesimal* geometry. This means that at each point of the manifold there exists a certain expression in the differentials of the variable coördinates which gives to

the first order of small quantities the distance between the given point and a point infinitesimally near it. This expression is a homogeneous quadratic form in the differentials of the variable coördinates. We naturally proceed to finite lengths, areas, and volumes by means of integration processes. The concept of the *geodesic curve*, or curve of shortest distance between two points, is of great importance in Riemann's geometry. These geodesic curves are generalizations of the straight lines of Euclidean geometry. They are defined within the manifold by means of differential equations formed by applying the calculus of variations. Another fundamental idea of Riemann's geometry is the concept of the *curvature* of the manifold at each point. We are familiar with the definition of the curvature of a curve from the elementary course in the calculus. This idea of curvature has been broadly generalized by Riemann for arbitrary manifolds. This idea of curvature is the scientific background for all of the statements about curved space and the non-existence of straight lines found during the last few years in newspaper accounts of Einstein's Relativity Theory.

Riemann's geometry includes as special cases all of the other metric geometries. It is strictly analytical in character. The geometries of Bolyai and Lobatchewsky, which were originally synthetic in character, can be realized in a Riemann space having a negative curvature. Euclid's geometry exists in a Riemann space having a zero curvature. By a *Riemann space* we mean the object associated with the homogeneous quadratic form mentioned in the preceding paragraph. This form is fundamental, since it leads to a definite system of measurement within the manifold. In fact the type of Riemann space is determined by the quadratic form.

Does Riemann's geometry have any physical significance? We may recall in this connection the statement of the famous German physicist, Max Planck: "What can be measured is what exists." This has become the guiding principle for modern physicists. It might be called their definition

of reality, and adherence to it demands that they admit in the structure of modern physical theory no quantity not susceptible to measurement. Perhaps the leader among these physicists is Albert Einstein. In seeking to realize this principle to the greatest degree he discovered the principles of the General Theory of Relativity. As are all theories in physics, it is mathematical in character, much more so perhaps than any other. It has stood successfully the supreme test of any scientific theory: the prediction of the occurrence of natural phenomena with precision. What has this to do with Riemannian geometry? Simply this: the geometry of Riemann and its technique, the tensor analysis, are the mathematical media for the formulation of the General Theory of Relativity.

Riemann's geometry has been greatly extended during the present century. Among the outstanding mathematicians engaged in this field of research are the Italian Tullio Levi-Civita, the German Hermann Weyl, and Oswald Veblen and what may be called the Princeton Group in this country. Those who are interested in further reading in this field may consult Veblen's "Invariants of Quadratic Differential Forms" (The Macmillan Company) and Levi-Civita's "The Absolute Differential Calculus." The latter work is an English translation of this author's work in Italian, and may be obtained through G. E. Stechert and Co., New York.

The question of the logical consistency of the geometries of Bolyai and Lobatchewsky was settled for all time by the great German mathematician Felix Klein in the year 1870. Essentially Klein set up a correspondence between these geometries and that of Euclid such that if there is any contradiction in the Euclidean geometry there will be a corresponding contradiction in these geometries and conversely. If we accept the Euclidean geometry as logically rigorous we must then accept the geometry of Bolyai and Lobatchewsky in the same sense. Klein also gave the classification of the various geometries current today: he named *hyperbolic* geometry the Bolyai-Lobatchewsky geometry, *parabolic* geometry the geometry of Euclid, and *elliptic*

geometry a special form of the geometry of Riemann. We must not forget, however, that Riemann's geometry is the general geometry.

3. *The Group Concept*

We cast a brief glance at the group concept because it is one of the oldest and deepest concepts of mathematics. According to Weyl, already mentioned above, a considerable part of group theory is implied in the art of surface ornamentation, in particular in the high development of this art realized by the ancient Egyptians. Weyl characterizes this as perhaps the original appearance of a mathematical idea in human culture. When we turn to the study of crystals in the field of physics we find that group theory is our mathematical tool for studying the structure of crystals. If we wish to study higher mechanics, such as is used in astronomy and the structure of the atom, we encounter the group concept. It is fundamental in geometry and algebra. We should now begin to appreciate the importance of this concept. We introduce it by means of two examples.

First we consider the motions of a rigid body M. We think of M as being displaced from the position A to the position B, and then to the position C. Thus the two displacements (AB) and (BC) can be thought of as equivalent to a single displacement (AC), or, in other words, any two displacements of a rigid body, or any number of successive displacements, can be replaced by a single displacement of the body. Again to any displacement (AB) there corresponds the inverse displacement (BA). The positions of the rigid body are represented by coördinates in some system of reference. The different positions are connected by some formal relation, or relations, between the coördinates. Such relations are called *transformations*. The motions of a rigid body form a typical example of a *group* of transformations. As we see from this example, any two successive transformations form another transformation of the group. The inverse of any transformation is again a transformation of the group. These are the essentials of the

definition of a group of transformations. It may be observed that the group of transformations, by reason of this definition, form a closed set of operations.

If we now consider a set of elements C of an arbitrary nature and an operation designated by the symbol $\circ$, we denote an operation performed on two elements a, b of C, in this order, by $a \circ b$. The set C is said to form a *group* relative to the operation $\circ$ if:

1. The result of the operation $a \circ b$ is an element of C.
2. The associative law is valid: $(a \circ b) \circ c = a \circ (b \circ c)$.
3. There exists within C an element i such that $a \circ i = i \circ a = a$.
4. For each element a of C there exists an element a' such that $a \circ a' = i$.

This is the general definition of a group. It is an abstraction from the idea of a transformation group, in which the elements of the transformation group have been replaced by elements of a completely arbitrary nature.

To illustrate this abstract definition we consider the set of the real numbers, and the operation of addition. Then, carrying out our illustration parallel to the general definition, we have:

1. The sum of two real numbers is again a real number: $a + b = c$.
2. The associative law holds: $(a + b) + c = a + (b + c)$.
3. If we take the number 0, $a + 0 = 0 + a = a$.
4. With each number a there is associated its negative a' such that $a + a' = 0$.

In this illustration the commutative law is also valid, but this law is not in general true in group operations. This simple illustration shows that we have been acquainted with the group concept ever since we learned the simplest operations in mathematics. It is to be hoped that we shall be sufficiently interested to investigate this topic further, and if so, we may look in "Projective Geometry" by Veblen

and Young, which will also strongly illuminate our mathematical ideas in general, and in the works on higher algebra by Professor L. E. Dickson. If these excite our interest sufficiently we can look at the group concept in mechanics, in the quantum theory of the atom, in crystallography, and in some of the higher branches of analysis.

4. *Analysis Situs*

Analysis situs, or position analysis, can be described as that branch of mathematics which treats of "figures" or geometric configurations, and in which shape and size are unimportant. Roughly speaking, the idea of proximity is the central idea in analysis situs. This idea necessarily involves the limit point of a set of points, since in this branch of mathematics we regard our figures as sets of points rather than as figures in the usual sense of the word. Let us consider a set of points, say a set defined in the Euclidean plane. If there exists within this set a point P such that in every neighborhood of P there is at least one point of the set, regardless of how small this neighborhood may be, then P is said to be a limit point of the set. A set may have no limit points. Again, every point of the set may be a limit point of the set. In analysis situs we are interested only in those properties of figures which can be expressed in terms of point sets and their limit points.

From another point of view analysis situs is interested only in those properties of figures or point sets which are not changed when the figures are subjected to a continuous or fluid motion. It is easy to conceive of a fluid motion carrying a circle into an ellipse, and this into a square, even though shape and area vary greatly from configuration to configuration. We can see what we mean by casting a loop of flexible thread on an eddying stream of water and observing the various shapes assumed by the loop. From the standpoint of analysis situs there is no essential difference between these various figures.

This idea of fluid motion is made exact by the idea of homeomorphism, which takes us back to the idea of the limit point. Consider two point sets S and S', between

which there has been set up a correspondence such that to each point of either set there corresponds one and only one point of the other set. Let A be any subset of S having a limit point P, and let B be the corresponding subset of S'. If the limit point P' of B corresponds to the limit point P of A, the sets S and S' are said to be *homeomorphic*. In other words, homeomorphism implies correspondence of the limit points of S and S'. Now we say that analysis situs is interested only in those properties of point sets which can be stated in terms of homeomorphisms. We may think that such a branch of mathematics is necessarily of short scope and barren of any usefulness outside itself, but if we attempt to pursue our studies of this subject this idea is soon dispelled. We might get a glimpse here of the possibilities in this field by remembering that from the standpoint of homeomorphism all circles, squares, ellipses, and polygons are equivalent.

For further information in this field of mathematics we refer the reader to the literature. We might mention the "Analysis of Matter" by Bertrand Russell as a work showing how analysis situs is used in the discussion of a physical problem. An extensive bibliography will be found at the end of the article in the Encyclopaedia Britannica. This latter remark applies to the other topics briefly discussed in this paper.

DOES THE PRESENT HIGH SCHOOL CURRICULUM IN MATHEMATICS PREPARE STUDENTS FOR COLLEGE?

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It is quite evident when one reads the literature on this topic that there has been very little accomplished in the way of obtaining definite results because of the many variables in the problem. The college professors think that they would be able to accomplish much if the students were not spoiled before they come to them. The secondary teachers know that it is the fault of either the elementary teachers or the college professors. They seem to be in an ideal situation to participate in the American pastime of "passing the buck," because they can pass it either up or down. The elementary teachers are to a great extent almost helpless because many parents do not have time to train their children properly, because of their numerous social duties. The parents claim that the teachers are hired to do this work and that good results should be obtained, even if the child is in the schoolroom only thirty hours each week. The results of this vicious circle are disastrous for the youth of the nation.

One of the most interesting studies is the report of the "Committee on Deans' Report," published in the *Proceedings of the Association of Colleges and Secondary Schools of the Southern States*. The purpose of this study was to test the efficiency of the secondary schools by the way their graduates succeeded in their freshman year in college. The names of those graduating in the spring and entering colleges in the fall were obtained from the principals and sent to the various registrars of the higher institutions for reports. These were tabulated according to states. Results for Texas in the years 1925–26, 1927–28, 1928–29, and 1929–30 have been summarized below in Table I. The percentage of failures in English, French, history, science, and

mathematics, and the average of the first four, are given for the entire state of Texas.

TABLE I*

Comparison of Percentages of Failures Among Texas College Freshmen in Mathematics and Four Other Major Fields

Year	(1) English	(2) French	(3) History	(4) Science	Average 1, 2, 3, 4	Mathematics
1925–26	16.3	14.2	13.5	18.8	13.9	24.6
1927–28	14.2	12.3	15.1	21.2	16.9	30.1
1928–29	13.9	17.6	14.2	21.4	15.6	25.3
1929–30	13.1	12.9	11.6	14.7	14.9	24.8
Average	14.4	14.3	13.6	19.0	15.3	26.2

*Proceedings of the Association of Colleges and Secondary Schools of the Southern States.

In analyzing these data it would appear that:

1. The number of failures in mathematics is considerably greater than in the other subjects.
2. Almost twice as many failures occur in mathematics as in English, French, and history.
3. 26.2 per cent failures in mathematics indicates a lack of preparation for college mathematics.

In the report for 1929–30, 16.8 per cent received a barely passing grade, which in most colleges is not considered as of graduating quality. That makes a total of 41.6 per cent who were unable to do satisfactory work in college mathematics during their first trial. No doubt some will contend that a portion of this is due to poor teaching in the institutions of higher learning. Dr. Joseph Seidlin completed a study[1] in 1931 entitled "A Critical Study of the Teaching of Elementary College Mathematics," in which this theory was held to be true. However, in a later article,[2] in the December issue of *The Mathematics Teacher* for 1932, he gave

[1]J. Seidlin, Bureau of Publications, Teachers College, Columbia University, New York, 1931.

[2]J. Seidlin, "The Contribution of High School Teaching to Ineffective College Teaching."

some very interesting by-products of the original investigation. He says that the following quotations "form a diversified and fairly complete indictment of present day high school teaching." A few striking ones are as follows:

1. "They (the students) come to us jaded, worn out, weary of mathematics."
2. "The most difficult task in teaching freshmen is this constant pulling, tugging at the strings of undeveloped or starved affection for elementary mathematics."
3. "The most commonly expressed sentiment is 'I understand it but I can't express it'."
4. "They lack ambition and initiative to generalize."
5. "They are happiest when a 'proof' (demonstration) is omitted."

The most interesting thing about these indictments is that they do not blame the secondary teachers for failure to give the students a mass of facts and involved manipulations of symbols, but rather for failure to inspire or to create an inner urge for investigation. Every college teacher knows that it is almost impossible for a student to make progress unless the desire to explore comes from within. A conference with a student having this inner urge will generally show that he has been closely associated with a teacher in the secondary school from whom he contracted the will to learn. Consequently the desire to investigate is perhaps the most important necessary condition for preparation for college. It is my opinion that secondary education in general has failed to produce graduates who have that important quality.

It is generally granted that mathematics is a tool subject for those who study physics and chemistry. Consequently reasons for failures in freshmen science might throw some light on the subject. In a recent interview with several teachers of college chemistry, all believed that an inadequate knowledge of elementary mathematics was responsible for numerous failures in freshman chemistry. They said that it was almost unbelievable how many high school graduates could divide one number by another and have no

idea where the decimal point should be placed; and that the difference between direct and inverse proportion can be explained by very few. No doubt many of these latter students could recite from *memory* the theorems about proportion in Book III of their plane geometry.

In the technical schools, such as those devoted to the various brands of engineering, there is a universal complaint against the preparatory training in algebra. It has become so prevalent that most engineering schools provide an elementary course for which no college credit is given. In many institutions this is known as "Mathematics 0." It is necessary for the student to have a thorough knowledge of the fundamentals of algebra as well as a natural ability to apply them in practical problems. Consequently, in view of the modern trend toward leaving out those things that are not useful at the secondary school age, I can see nothing but trouble ahead unless the technical institutions of higher learning bridge the gap with preparatory courses.

In answering the question, "Does the present high school curriculum in mathematics prepare students for college?" it is my opinion that it is not accomplishing as much as it should. First, there are too many failures in freshman mathematics in college; secondly, the students choosing science in institutions of higher learning are handicapped because of their deficiencies in arithmetic; thirdly, the students themselves are "fed up" with mathematics, so that one of their most important tools for all scientific work is about as useful as an electric fan at the north pole.

Perhaps it might be best to suggest several ways in which our secondary education might become more efficient. I shall enumerate several with brief comments:

1. A reorganization of junior and senior high school mathematics so that it will be useful, inspirational, and at the same time contain sufficient content to be an aid to scientific studies. Because of the proposed change by the State Board of Education in the mathematics curriculum of the secondary schools, this suggestion should be considered

very carfully. Several universities, including The University of Texas, Southern Methodist University, Texas Christian University, and the various state teachers' colleges, require only two units of high school mathematics for entrance, and grant an A.B. degree in certain fields without additional university mathematics. It is the opinion of the State Board of Education that through a reorganization of the curriculum in mathematics in the junior as well as in the senior high school, the student would have as good a working knowledge in mathematics after two years as at present after three. In fact, several special committees chosen from the secondary schools of Texas are investigating this particular problem.

2. Teachers of secondary mathematics should be properly trained in mathematics as well as in education. College teachers should recommend for mathematics positions in the secondary schools only those who have had at least two years of college mathematics and preferably three. In a recent issue of *The Mathematics Teacher,* Professor Hedrick, of the University of California, reports that there are states in the North where one-third of all the teachers of mathematics who hold a college degree never took a single hour of college mathematics. It is easy to see that in most of these cases no far-reaching inspiration could be imparted to the students. But this is not the fault of the teachers. The responsibility rests squarely with the administrators for sponsoring such a condition.

3. A change should be made to lessen the number of hours of education and to require at least three years of successful teaching experience before receiving a permanent certificate to teach in Texas. At present the requirements are a few hours of practice teaching and twenty-four hours of education in addition to the requirements for a college degree. This certificate entitles the holder to teach *any or all of the entire curriculum* in the state of Texas, even though he may never have studied the particular subject taught. The Committee on "Required Courses in Education" reported at the December meeting of the American

Association of University Professors the following interesting conclusion:

> A maximum of twelve semester hours is ample to cover that part of professional training which can be regarded as essential for the beginning teacher who has a bachelor's degree from a standard college or university, and who qualifies for teaching an academic subject. The training should involve practice teaching and methods, the methods course being closely integrated with the practice teaching. Courses in psychology or educational psychology, when these are required, should be counted towards the requirement.

4. A closer coöperation should be established between secondary and college teachers. If one of them has a serious problem, it automatically affects the other. Consequently they need to study the problems together instead of continually "passing the buck."

THE BROWN UNIVERSITY PRIZE EXAMINATION

In 1933 the Brown University Entrance Prize Examination for "bona fide" Freshmen was given as usual on the second Saturday in October. More than sixty students presented themselves for this contest. The winners of the three prizes are as follows:

First prize, George Chesnut, from the Oak Cliff High School, Dallas;

Second prize, Jerry McAfee, an engineering student, from the Port Arthur High School;

Third prize, Frank L. Woodbury, from the Stripling High School, Fort Worth.

It is worthy of note that each of the successful contestants was in the first quartile of the high school from which he graduated.

The following four problems constituted the examination:

1. Three circles are tangent externally at points A, B, C. The chords AB and AC are produced to cut the circle BC at D and E. Prove that DE is a diameter.

2. Compute the portion of the area of a circle included between two parallel chords, one equal to the radius and the other to the side of the inscribed equilateral triangle.

3. The sum of two numbers is s, and the sum of the quotients of A by the two numbers is also s. Show that the quadratic equation found in order to determine one number has for its other root the second number.

4. Show that if a, b, c, d and n are positive integers and if

$$\frac{a}{b} < \frac{c}{d},$$

then

$$\frac{a}{b} < \frac{a+nc}{b+nd} < \frac{c}{d}$$